ON GRADED LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

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Abstract. Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a standard graded ring, M be a finite graded R-module and J be a homogenous ideal of R. In this paper we study the graded structure of the i-th local cohomology module of M defined by a pair of ideals (R_+, J), i.e. $H^i_{R_+,J}(M)$. More precisely, we discuss finiteness property and vanishing of the graded components $H^i_{R_+,J}(M)_n$.

Also, we study the Artinian property and tameness of certain submodules and quotient modules of $H^i_{R_+,J}(M)$.

1. introduction

Let R denotes a commutative Noetherian ring, M an R-module and I and J stand for two ideals of R. Takahashi, et. all in [10] introduced the i-th local cohomology functor with respect to (I, J), denoted by $H^i_{I,J}(-)$, as the i-th right derived functor of the (I, J)-torsion functor $\Gamma_{I,J}(-)$, where $\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}$. This notion is the ordinary local cohomology functor when $J = 0$ (see [3]). The main motivation for this generalization comes from the study of a dual of ordinary local cohomology modules $H^i_I(M)$ (see [9]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [10], [4] and [5].

Now, let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a standard graded Noetherian ring, i.e. the base ring R_0 is a commutative Noetherian ring and R is generated, as an R_0-algebra, by finitely many elements of R_1. Also, let J be a homogenous ideal of R, M be a graded R-module and $R_+ := \oplus_{n\in\mathbb{N}} R_n$ be the irrelevant ideal of R. It is well known ([3, Section 12]) that for all $i \geq 0$ the i-th local cohomology module $H^i_J(M)$ of M with respect to J has a natural grading

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and that, in the case where M is finite, $H^i_{R_+}(M)_n$ is a finite R_0-module for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$ ([3, Theorem 15.1.5]).

In this paper, first, we show that $H^i_{I,J}(M)$ has a natural grading, when I and J are homogenous ideals of R and M is a graded R-module. Then, we show that, although in spite of the ordinary case, $H^i_{R_+,J}(M)_n$ might be non-finite over R_0 for some $n \in \mathbb{Z}$ and non-zero for all $n \gg 0$, but in some special cases they are finite for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$. More precisely, we show that if $(R_0, \mathfrak{m}_0)$ is local, $R_+ \subseteq \mathfrak{b}$ is an ideal of R and $\bigcap\limits_{k=0}^{\infty} \mathfrak{m}_0^k H^i_{\mathfrak{b},\mathfrak{m}_0 R}(M)_n = 0$ for all $n \gg 0$, then $H^i_{\mathfrak{b},\mathfrak{m}_0 R}(M)_n = 0$ for all $n \gg 0$. Also, we present an equivalent condition for the finiteness of components $H^i_{R_+,J}(M)_n$ (Theorem 3.3).

In the last section, first, we study the asymptotic stability of the set $\{Ass_{R_0}(H^i_{R_+,J}(M)_n)\}_{n\in\mathbb{Z}}$ for $n \to -\infty$ in a special case (Theorem 4.1). Then we present some results about Artinianness of some quotients of $H^i_{R_+,J}(M)$. In particular, we show that if R_0 is a local ring with maximal ideal $\mathfrak{m}_0$ and $c \in \mathbb{Z}$ such that $H^i_{R_+,\mathfrak{m}_0 R}(M)$ is Artinian for all $i > c$, then the R-module $H^c_{R_+,\mathfrak{m}_0 R}(M)/\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)$ is Artinian (Theorem 4.2). Finally, we show that $H^i_{R_+,J}(M)$ is "tame" in a special case (Corollary 4.4).

2. Graded local cohomology modules defined by a pair of ideals

Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a graded ring, I and J be two homogenous ideals of R and $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded R-module. Then, it is natural to ask whether the local cohomology modules $H^i_{I,J}(M)$ for all $i \in \mathbb{N}_0$, also carry structures as graded R-modules. In this section we show that it has affirmative answer.

First we show that, the (I, J)-torsion functor $\Gamma_{I,J}(-)$ can be viewed as a (left exact, additive) functor from ${}^*\mathcal{C}$ to itself. Since the category ${}^*\mathcal{C}$, of all graded R-modules and homogeneous R-homomorphisms, is an Abelian category which has enough projective object and enough injective objects, we can therefore carry out standard techniques of homological algebra in this category. Hence, we can form the right derived functors ${}^*H^i_{I,J}(-)$ of $\Gamma_{I,J}(-)$ on the category ${}^*\mathcal{C}$.

Lemma 2.1. *Let $x = x_{i_1} + \cdots + x_{i_k} \in M$ be such that $x_{i_j} \in M_{i_j}$ for all $j = 1, \cdots, k$. Then*

$$r(Ann(x)) = \bigcap_{j=1}^{k} r(Ann(x_{i_j}))$$

Proof. $\supseteq$: Is clear.

$\subseteq$: First we show that if $y = y_{l_1}+\cdots+y_{l_m} \in Ann(x)$ such that $y_{l_k} \in R_{l_k}$ for all $k = 1, \cdots, m$ and $l_1 < l_2 < \cdots < l_m$, then $y_{l_1} \in \cap_{j=1}^{k} r(Ann(x_{i_j}))$. We have

$$0 = yx = \sum_{j=1}^{n}\sum_{k=1}^{m} y_{l_k} x_{i_j}, \qquad (*)$$

comparing degrees, we get $y_{l_1}x_{i_1} = 0$. Let $j_0 > 1$ and suppose, inductively, that for all $j' < j_0$, $y_{l_1}^{j'} x_{i_{j'}} = 0$. Then using $(*)$ we get

$$\sum_{j=1}^{n}\sum_{k=1}^{m} y_{l_k} y_{l_1}^{j_0-1} x_{i_j} = 0.$$

Again, comparing degrees, we have $y_{l_1}^{j_0} x_{i_{j_0}} = 0$. So, $y_{l_1} \in \cap_{j=1}^{k} r(Ann(x_{i_j}))$.

Now, let $y = y_{l_1} + \cdots + y_{l_m} \in r(Ann(x))$ such that $l_1 < l_2 < \ldots < l_m$ and $y_{l_j} \in R_{l_j}$ for all $j = 1, \cdots, m$. Then, there exists $s \in \mathbb{N}_0$ such that $y^s x = 0$.

In order to show that $y \in \cap_{j=1}^{k} r(Ann(x_{i_j}))$ we proceed by induction on m. The result is clear in the case $m = 1$. Now, suppose inductively that $m > 1$ and the result has been proved for values less than m. Using the above agreement, we know that $y_{l_1} \in \bigcap\limits_{j=1}^{k} r(Ann(x_{i_j})) \subseteq r(Ann(x))$. Then $y_{l_2} + \cdots + y_{l_m} = y - y_{l_1} \in r(Ann(x))$. By inductive hypothesis, $y_{l_2} + \cdots + y_{l_m} \in \bigcap\limits_{j=1}^{k} r(Ann(x_{i_j}))$ and so, $y \in \bigcap\limits_{j=1}^{k} r(Ann(x_{i_j}))$.

□

Lemma 2.2. *$\Gamma_{I,J}(M)$ is a graded R-module.*

Proof. Let $x \in \Gamma_{I,J}(M)$. Assume that $x = x_{i_1} + \cdots + x_{i_k}$ where for all $j = 1, 2, \cdots, k$, $x_{i_j} \in M_{i_j}$ and $i_1 < i_2 < \cdots < i_k$. We show that $x_{i_1}, \cdots, x_{i_k} \in \Gamma_{I,J}(M)$. Since R is Noetherian, there is $t' \in \mathbb{N}$ such that $(r(Annx_{i_j}))^{t'} \subseteq Ann(x_{i_j})$ for all $j = 1, 2, \cdots, k$. Let $n \in \mathbb{N}_0$ be such that $I^n \subseteq Ann(x) + J$. So, by Lemma 2.1, for all $j = 1, 2, \cdots, k$ we have

$$I^{2nt'} \subseteq (Ann(x) + J)^{2t'} \subseteq (Ann(x))^{t'} + J^{t'} \subseteq (r(Ann(x)))^{t'} + J^{t'} = (\cap_{j=1}^{k} r(Ann(x_{i_j})))^{t'} + J^{t'}$$
$$\subseteq Ann(x_{i_j}) + J.$$

Thus $x_{i_j} \in \Gamma_{I,J}(M)$, as required. □

To calculate graded local cohomology module $^*H^i_{I,J}(M)$ $(i \in \mathbb{N}_0)$, one proceeds as follows: Taking an *injective resolution

$$E^{\bullet} : 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \cdots \to E^i \xrightarrow{d^i} E^{i+1} \to \cdots,$$

of M in $^*\,\mathcal{C}$, applying the functor $\Gamma_{I,J}(-)$ to it and taking the i-th cohomology module of this complex, we get

$$\frac{ker\Gamma_{I,J}(d^i)}{im\Gamma_{I,J}(d^{i-1})}$$

which is denoted by $^*\ H^i_{I,J}(M)$ and is a graded R-module.

Remark 2.3. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence of R-modules and R-homomorphisms. Then, for each $i \in \mathbb{N}_0$, there is a homogeneous connecting homomorphism $^*\ H^i_{I,J}(N) \to^*\ H^{i+1}_{I,J}(L)$ and these connecting homomorphisms make the resulting homogenous long exact sequence*

$$\begin{array}{llll}
0 \to^*\ H^0_{I,J}(L) & \xrightarrow{^*\ H^0_{I,J}(f)} {}^*\ H^0_{I,J}(M) & \xrightarrow{^*\ H^0_{I,J}(g)} {}^*\ H^0_{I,J}(N) \\
\to^*\ H^1_{I,J}(L) & \xrightarrow{^*\ H^1_{I,J}(f)} {}^*\ H^1_{I,J}(M) & \xrightarrow{^*\ H^1_{I,J}(g)} {}^*\ H^1_{I,J}(N) \\
\to \cdots & & \\
\to^*\ H^i_{I,J}(L) & \xrightarrow{^*\ H^i_{I,J}(f)} {}^*\ H^i_{I,J}(M) & \xrightarrow{^*\ H^i_{I,J}(g)} {}^*\ H^i_{I,J}(N) \\
\to^*\ H^{i+1}_{I,J}(L) \to \cdots . & &
\end{array}$$

The reader should also be aware of the 'natural' or 'functorial' properties of these long exact sequences.

Definition 2.4. *We define a partial order on the set*

$$^*\ \widetilde{W}(I,J) := \{\mathfrak{a} : \mathfrak{a} \text{ is a homogenous ideal of } R; I^n \subseteq \mathfrak{a} + J \text{ for some integer } n \geq 1\},$$

by letting $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{a} \supseteq \mathfrak{b}$, for $\mathfrak{a}, \mathfrak{b} \in^*\ \widetilde{W}(I,J)$. If $\mathfrak{a} \leq \mathfrak{b}$, then we have $\Gamma_{\mathfrak{a}}(M) \subseteq \Gamma_{\mathfrak{b}}(M)$. Therefore, the relation $\leq$ on $^*\ \widetilde{W}(I,J)$ with together the inclusion maps make $\{\Gamma_{\mathfrak{a}}(M)\}_{\mathfrak{a} \in^*\ \widetilde{W}(I,J)}$ into a direct system of graded R-modules.

As Takahashi et. all in [10] showed the relation between the local cohomology functor $H^i_I(-)$ and $H^i_{I,J}(-)$, we show the same relation between graded version of them as follows.

Proposition 2.5. *Let M be a graded R-module. Then there is a natural graded isomorphism*

$$\big({}^* \ H^i_{I,J}(M)\big)_{i\in\mathbb{N}_0} \cong \big(\varinjlim_{\mathfrak{a}\in^* \ \widetilde{W}(I,J)} {}^* \ H^i_{\mathfrak{a}}(M)\big)_{i\in\mathbb{N}_0},$$

of strongly connected sequences of covariant functors.

Proof. First of all, we show that $\Gamma_{I,J}(M) = \bigcup_{\mathfrak{a}\in^* \ \widetilde{W}(I,J)} \Gamma_{\mathfrak{a}}(M)$.

$\supseteq$: Suppose that $x \in \bigcup_{\mathfrak{a}\in^* \ \widetilde{W}(I,J)} \Gamma_{\mathfrak{a}}(M)$. Then there are $\mathfrak{a} \in^* \ \widetilde{W}(I,J)$ and integer n such that $I^n \subseteq \mathfrak{a} + J$ and $x \in \Gamma_{\mathfrak{a}}(M)$. Let $t \in \mathbb{N}_0$ be such that $\mathfrak{a}^t \subseteq Ann(x)$. Therefore $I^{2nt} \subseteq (\mathfrak{a} + J)^{2t} \subseteq \mathfrak{a}^t + J^t \subseteq Ann(x) + J$ and so $x \in \Gamma_{I,J}(M)$.

$\subseteq$: Conversely, let $x \in \Gamma_{I,J}(M)$. Then $I^n \subseteq Ann(x) + J$ for some $n \in \mathbb{N}$. We show that $x \in \Gamma_{\mathfrak{a}}(M)$ such that $\mathfrak{a} = r(Ann(x))$. As $r(Ann(x))$ is homogenous by Lemma 2.1, and $I^n \subseteq Ann(x) + J$, we have $r(Ann(x)) \in^* \ \widetilde{W}(I,J)$ and $x \in \Gamma_{r(Ann(x))}(M)$.

Now, [3, Exercise 12.1.7] implies the desired isomorpism.

□

Remark 2.6. *If we forget the grading on* ${}^* \ H^i_{I,J}(M)$*, the resulting* R*-module is isomorphic to* $H^i_{I,J}(M)$*. More precisely, using* [3, Proposition 12.1.3] *and the fact that the direct systems* $\widetilde{W}(I,J)$ *and* ${}^* \ \widetilde{W}(I,J)$ *are cofinal we have*

$$H^i_{I,J}(E) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(E) \cong \varinjlim_{\mathfrak{a}\in^* \ \widetilde{W}(I,J)} H^i_{\mathfrak{a}}(E) = 0,$$

for all $i > 0$ *and all* **injective* R*-module* E*. Now, using similar argument as used in* [3, Corollary 12.3.3]*, one can see that there exists an equivalent of functors*

$$H^i_{I,J}(-\rceil_{^*\mathcal{C}}) \cong^* \ H^i_{I,J}(-)$$

from ${}^* \ \mathcal{C}$ *to itself.*

As a consequence of the above remark and [3, Remark 13.1.9(ii)], we have the following.

Corollary 2.7. *Let* $t \in \mathbb{Z}$*, then*

$$H^i_{I,J}(M(t)) \cong (H^i_{I,J}(M))(t),$$

for all $i \in \mathbb{N}$*, where* $(.)(t) :^* \ \mathcal{C} \to^* \ \mathcal{C}$ *is the* t*-th shift functor.*

3. VANISHING AND FINITENESS OF COMPONENTS

A crucial role in the study of the graded local cohomology is vanishing and finiteness of their components. As one can see in Theorem 15.1.5 [3], $H^i_{R_+}(M)_n$ is a finite R_0-module for all $n \in \mathbb{Z}$ and it vanishes for all $n \gg 0$. In this section we show that, although it is not the same for $H^i_{R_+,J}(M)$, but it holds in some special cases.

In the rest of this paper, we assume that $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a standard graded ring and M is a finite graded R-module.

Local cohomology with respect to a pair of ideals does not satisfy in Theorem 15.1.5 [3], in general, as the following counterexample shows.

Remark 3.1. *(i) Let $R = \mathbb{Z}[X]$ and $R_+ = (X)$. We can see that $\Gamma_{R_+,R_+}(\mathbb{Z}[X])_n = \mathbb{Z}[X]_n \neq 0$ for all $n \in \mathbb{N}_0$.*

(ii) Assume that J is an ideal of R generated by elements of degree zero such that $JR_+ = 0$. It is easy to see that in this condition $\Gamma_{R_+,J}(M) = \Gamma_{R_+}(M)$ and therefore, [3, Theorem 15.1.5] *holds for $H^i_{R_+,J}(M)$.*

(iii) Let $(R_0, \mathfrak{m}_0)$ be a local ring and $\dim R_0 = 0$. Then $\Gamma_{R_+,\mathfrak{m}_0 R}(M) = \Gamma_{R_+}(M)$ and, again, [3, Theorem 15.1.5] *holds for $H^i_{R_+,\mathfrak{m}_0 R}(M)$.*

The following proposition, indicates a vanishing property on the graded components of $H^i_{\mathfrak{b},\mathfrak{m}_0 R}(M)$ for ideal $\mathfrak{b} = \mathfrak{b}_0 + R_+$ where $\mathfrak{b}_0$ is an ideal of R_0 and $\mathfrak{m}_0$ is the unique maximal ideal of R_0. Vanishing of the components $H^i_{\mathfrak{b}}(M)_n$ for $n \gg 0$ has already been studied in [7].

Theorem 3.2. *Assume that $(R_0, \mathfrak{m}_0)$ is local and $i \in \mathbb{N}_0$. Let $\mathfrak{b} := \mathfrak{b}_0 + R_+$ where $\mathfrak{b}_0$ is an ideal of R_0 such that for all finite graded R-module M, $\bigcap_{k=0}^{\infty} \mathfrak{m}_0^k H^i_{\mathfrak{b},\mathfrak{m}_0 R}(M)_n = 0$ for all $n \gg 0$. Then $H^i_{\mathfrak{b},\mathfrak{m}_0 R}(M)_n = 0$ for all $n \gg 0$ and all finite graded R-module M.*

Proof. We proceed by induction on $\dim M$.

Let $J := \mathfrak{m}_0 R$. If $\dim M = 0$, then using [8, Theorem 1] $\Gamma_{\mathfrak{b},J}(M)_n = M_n = 0$ for all $n \gg 0$.

Now, let $\dim M > 0$. Considering the long exact sequence

$$H^i_{\mathfrak{b},J}(\Gamma_J(M))_n \to H^i_{\mathfrak{b},J}(M)_n \to H^i_{\mathfrak{b},J}(\overline{M})_n \to H^{i+1}_{\mathfrak{b},J}(\Gamma_J(M))_n,$$

where $\overline{M} = M/\Gamma_J M$, by [7, Proposition 1.1] we get $H^i_{\mathfrak{b},J}(M)_n \cong H^i_{\mathfrak{b},J}(\overline{M})_n$ for all $n \gg 0$. Therefore, we may assume that M is J-torsion free and so there exists $x_0 \in \mathfrak{m}_0 \backslash Z_{R_0}(M)$.

Now, the exact sequence

$$0 \to M \xrightarrow{x_0} M \to M/x_0M \to 0$$

implies the exact sequence

$$H^i_{\mathfrak{b},J}(M)_n \xrightarrow{x_0} H^i_{\mathfrak{b},J}(M)_n \to H^i_{\mathfrak{b},J}(M/x_0M)_n.$$

Then, by the assumptions and the inductive hypothesis,

$$H^i_{\mathfrak{b},J}(M/x_0M)_n = 0$$

for all $n \gg 0$. So,

$$H^i_{\mathfrak{b},J}(M)_n = x_0 H^i_{\mathfrak{b},J}(M)_n$$

for all $n \gg 0$. Therefore,

$$H^i_{\mathfrak{b},J}(M)_n = \bigcap_{k=0}^{\infty} x_0^k H^i_{\mathfrak{b},J}(M)_n = 0$$

for all $n \gg 0$. Now, the result follows by induction.

□

In the following we present an equivalent condition for the finiteness of components $H^i_{R_+,J}(M)_n$.

Theorem 3.3. *Let $(R_0, \mathfrak{m}_0)$ be local and $J_0 \subseteq \mathfrak{m}_0$ be an ideal of R_0. Then the following statements are equivalent.*

a) For all finite graded R-module M and all $i \in \mathbb{N}_0$, $H^i_{R_+,J_0R}(M)_n = 0$ for $n \gg 0$.
b) For all finite graded R-module M, all $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, $H^i_{R_+,J_0R}(M)_n$ is a finite R_0-module.

Proof. Let $J = J_0R$.

a)$\Rightarrow$ b) Let M be a non-zero finite graded R-module. We proceed by induction on i. It is clear that $H^0_{R_+,J}(M)$ is a finite R-module and then $H^0_{R_+,J}(M)_n$ is finite as an R_0-module for all $n \in \mathbb{Z}$.

Now, suppose that $i > 0$ and the result is proved for smaller values than i. As $H^i_{R_+,J}(M) \cong H^i_{R_+,J}(M/\Gamma_{R_+,J}(M))$, we may assume that M is an (R_+, J)-torsion free R-module and so R_+-torsion free R-module. Hence R_+ contains a non zero-divisor on M. As $M \neq R_+M$, there exists a homogeneous element $x \in R_+$ of degree t, which is a non zero-divisor on M,

by [3, Lemma 15.1.4]. We use the exact sequence $0 \to M \xrightarrow{x} M(t) \to (M/xM)(t) \to 0$ of graded R-modules and homogeneous homomorphisms to obtain the exact sequence

$$H^{i-1}_{R_+,J}(M/xM)_{n+t} \to H^i_{R_+,J}(M)_n \xrightarrow{x} H^i_{R_+,J}(M)_{n+t}$$

for all $n \in \mathbb{Z}$. It follows from the inductive hypothesis that $H^{i-1}_{R_+,J}(M/xM)_j$ is a finite R_0-module for all $j \in \mathbb{Z}$. Let $s \in \mathbb{Z}$ be such that $H^i_{R_+,J}(M)_m = 0$ for all $m \geq s$. Fix an integer n, then for some $k \in \mathbb{N}_0$ we get $n + kt \geq s$ and then $H^i_{R_+,J}(M)_{n+kt} = 0$. Now, for all $j = 0, \cdots, k-1$, we have the exact sequence

$$H^{i-1}_{R_+,J}(M/xM)_{n+(j+1)t} \to H^i_{R_+,J}(M)_{n+jt} \xrightarrow{x} H^i_{R_+,J}(M)_{n+(j+1)t}.$$

Since $H^i_{R_+,J}(M)_{n+kt} = 0$ and $H^{i-1}_{R_+,J}(M/xM)_{n+kt}$ is a finite R_0-module, so $H^i_{R_+,J}(M)_{n+(k-1)t}$ is a finite R_0-module. Therefore $H^i_{R_+,J}(M)_{n+jt}$ is a finite R_0-module for $j = 0, \ldots, k-1$. Now, the result follows by induction.

b)$\Rightarrow$ a) The result follows from the above theorem.

$\square$

4. Asymptotic behavior of $H^i_{R_+,J}(M)_n$ for $n \ll 0$

In this section we consider the asymptotic behavior of components $H^i_{R_+,J}(M)_n$ when $n \to -\infty$. More precisely, first we study the asymptotic stablity of the set $\{Ass_{R_0}(H^i_{R_+,J}(M)_n)\}_{n\in\mathbb{Z}}$ in a special case. Then, we investigate the Artinianness and tameness of some quotients and submodules of $H^i_{R_+,J}(M)$.

Let us recall that for a given sequence $\{S_n\}_{n\in\mathbb{Z}}$ of sets $S_n \subseteq Spec(R_0)$, we say that $\{S_n\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n \to -\infty$, if there is some $n_0 \in \mathbb{Z}$ such that $S_n = S_{n_0}$ for all $n \leq n_0$ (see [1]). Let the base ring R_0 be local and $i \in \mathbb{N}_0$ be such that the R-module $H^j_{R_+}(M)$ is finite for all $j < i$. In [2, Lemma 5.4] it has been shown that $\{\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n \to -\infty$. The next theorem use similar argument to improve this result to local cohomology modules defines by a pair of ideals.

Theorem 4.1. *Let $(R_0, \mathfrak{m}_0)$ be a local ring with infinite residue field and $i \in \mathbb{N}_0$ be such that the R-module $H^j_{R_+,J}(M)$ is finite for all $j < i$. If one of the equivalent conditions of the Theorem 3.3 holds, then $Ass_{R_0}(H^i_{R_+,J}(M)_n)$ is asymptotically stable for $n \to -\infty$.*

Proof. We use induction on i. For $i = 0$ the result is clear from the fact that $H^0_{R_+,J}(M)_n = 0$ for all $n \ll 0$. Now, let $i > 0$. In view of the natural graded isomorphism, $H^i_{R_+,J}(M) \cong$

$H^i_{R_+,J}(M/\Gamma_{R_+,J}(M))$, for all $i \in \mathbb{N}_0$, and using [3, Lemma 15.1.4], we may assume that there exists a homogeneous element $x \in R_1$ which is a non zero-divisor on M. Now, by the long exact sequence

$$H^{j-1}_{R_+,J}(M) \to H^{j-1}_{R_+,J}(M/xM) \to H^j_{R_+,J}(M)(-1) \xrightarrow{x} H^j_{R_+,J}(M)$$

for all $j \in \mathbb{Z}$, we have $H^j_{R_+,J}(M/xM)$ is finite for all $j < i-1$. Hence, by the inductive hypothesis,

$$\mathrm{Ass}_{R_0}(H^{i-1}_{R_+,J}(M/xM)_n) = \mathrm{Ass}_{R_0}(H^{i-1}_{R_+,J}(M/xM)_{n_1}) =: X$$

for some $n_1 \in \mathbb{Z}$ and all $n \leq n_1$. Furthermore, there is some $n_2 < n_1$ such that $H^{i-1}_{R_+,J}(M)_{n+1} = 0$ for all $n \leq n_2$. Then for all $n \leq n_2$ we have the exact sequence

$$0 \to H^{i-1}_{R_+,J}(M/xM)_{n+1} \to H^i_{R_+,J}(M)_n \xrightarrow{x} H^i_{R_+,J}(M)_{n+1}.$$

Thus, it shows that

$$X \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n) \subseteq X \cup \mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_{n+1})$$

for all $n \leq n_2$. Hence

$$\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n) \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_{n+1})$$

for all $n < n_2$ and, using the assumption, the proof is complete.

□

In the rest of paper, we pay attention to the Artinianness property of the graded modules $H^i_{R_+,J}(M)$. The following proposition, gives a graded analogue of [5, Theorem 2.2].

Theorem 4.2. *Assume that R_0 is a local ring with maximal ideal $\mathfrak{m}_0$. If $c \in \mathbb{Z}$ and $H^i_{R_+,\mathfrak{m}_0 R}(M)$ is Artinian for all $i > c$, then the R-module $H^c_{R_+,\mathfrak{m}_0 R}(M)/\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)$ is Artinian.*

Proof. Let $\mathfrak{m} := \mathfrak{m}_0 + R_+$ be the unique graded maximal ideal of R and let $J := \mathfrak{m}_0 R$. We have $H^i_{R_+,J}(M) = H^i_{\mathfrak{m},J}(M)$ for all i. Thus we can replace R_+ by $\mathfrak{m}$. We proceed the assertion by induction on $n := \dim M$. The result is clear in the case $n = 0$. Let $n > 0$ and that the statement is proved for all values less than n. Now, using the long exact sequence

$$H^i_{\mathfrak{m},J}(\Gamma_{\mathrm{J}}(M) \to H^i_{\mathfrak{m},J}(M) \to H^i_{\mathfrak{m},J}(M/\Gamma_J(M)) \to H^{i+1}_{\mathfrak{m},J}(\Gamma_{\mathrm{J}}(M)),$$

and the fact that $H^i_{\mathfrak{m},J}(\Gamma_J(M)) = H^i_{\mathfrak{m}}(\Gamma_J(M))$ is Artinian for all i, replacing M with $M/\Gamma_J(M)$, we may assume that $\Gamma_J(M) = 0$. Therefore, there exists $x_0 \in \mathfrak{m}_0 \backslash Z_{R_0}(M)$. Now, the long exact sequence

$$H^i_{\mathfrak{m},J}(M) \xrightarrow{x_0} H^i_{\mathfrak{m},J}(M) \xrightarrow{\alpha_i} H^i_{\mathfrak{m},J}(M/x_0M) \xrightarrow{\beta_i} H^{i+1}_{\mathfrak{m},J}(M)$$

implies that $H^i_{\mathfrak{m},J}(M/x_0M)$ is Artinian for all $i > c$ and so, by inductive hypothesis, $H^c_{\mathfrak{m},J}(M/x_0M)/\mathfrak{m}_0H^c_{\mathfrak{m},J}(M/x_0M)$ is Artinian. Considering the exact sequences

$$0 \to Im\alpha_c \to H^c_{\mathfrak{m},J}(M/x_0M) \to Im\beta_c \to 0$$

and

$$H^c_{\mathfrak{m},J}(M) \xrightarrow{x_0} H^c_{\mathfrak{m},J}(M) \xrightarrow{\alpha_c} Im\alpha_c \to 0,$$

we get the following exact sequences

$$Tor^R_1(R_0/\mathfrak{m}_0, Im\beta_c) \to Im\alpha_c/\mathfrak{m}_0Im\alpha \to H^c_{\mathfrak{m},J}(M/x_0M)/\mathfrak{m}_0H^c_{\mathfrak{m},J}(M/x_0M) \qquad (A)$$

and

$$H^c_{\mathfrak{m},J}(M)/\mathfrak{m}_0H^c_{\mathfrak{m},J}(M) \xrightarrow{x_0} H^c_{\mathfrak{m},J}(M)/\mathfrak{m}_0H^c_{\mathfrak{m},J}(M) \to Im\alpha_c/\mathfrak{m}_0Im\alpha_c \to 0. \qquad (B)$$

these two exact sequences implies that $H^c_{\mathfrak{m},J}(M)/\mathfrak{m}_0H^c_{\mathfrak{m},J}(M)$ is Artinian and the assertion follows. □

Let I, J be ideals of R. Chu and Wang in [5] defined $cd(I, J, R) := \sup\{i; H^i_{I,J}(M) \neq 0\}$.

The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.3. *Assume that R_0 is local with maximal ideal $\mathfrak{m}_0$. If $c := cd(R_+, \mathfrak{m}_0R, M)$. Then $H^c_{R_+,\mathfrak{m}_0R}(M)/\mathfrak{m}_0H^c_{R_+,\mathfrak{m}_0R}(M)$ is Artinian.*

Let $T = \bigoplus_{n\in\mathbb{N}_0} T_n$ be a graded R-module. Following [1], we say that T is tame or asymptotically gap free if either $T_n = 0$ for all $n \ll 0$ or else $T_n \neq 0$ for all $n \ll 0$. Now, as an application of the above Corollary, we have the following.

Corollary 4.4. *Let $(R_0, \mathfrak{m}_0)$ be local and $c := cd(R_+, \mathfrak{m}_0R, M)$. If one of the equivalent conditions of Theorem 3.3 holds, then $H^c_{R_+,\mathfrak{m}_0R}(M)$ is tame.*

Proof. Let $J = \mathfrak{m}_0R$. Since $H^c_{R_+,\mathfrak{m}_0R}(M)/\mathfrak{m}_0H^c_{R_+,\mathfrak{m}_0R}(M)$ is Artinian so it is tame. Now, the result follows using Nakayama's lemma. □

Proposition 4.5. *Let* $(R_0, \mathfrak{m}_0)$ *be local,* $J \subseteq R_+$ *be a homogenous ideal of* R *and* $g(M) := \sup\{i : \forall j < i, \ell_{R_0}(H^j_{R_+}(M)_n) < \infty, \forall n \ll 0\}$ *be finite. Then, the graded* R*-module* $H^i_{\mathfrak{m}_0 R, J}(H^j_{R_+}(M))$ *is Artinian for* $i = 0, 1$ *and all* $j \leq g(M)$.

Proof. Since $J \subseteq R_+$, so $H^j_{R_+}(M)$ is J-torsion. Therefore, $H^i_{\mathfrak{m}_0 R, J}(H^j_{R_+}(M)) \cong H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))$. Now, the result follows from [6, Theorem 2.4]. □

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